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On spectral density of Neumann matrices

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Abstract

In hep-th/0111281 the complete set of eigenvectors and eigenvalues of Neumann matrices was found. It was shown also that the spectral density contains a divergent constant piece that being regulated by truncation at level L equals $\frac{\log L}{2\pi}$. In this paper we find an exact analytic expression for the finite part of the spectral density. This function allows one to calculate finite parts of various determinants arising in string field theory computations. We put our result to some consistency checks.

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1 Introduction

The basic ingredient in the construction of the covariant string field theory is Witten's star product [1]. In the algebraic formulation this star product is specified by a string three-vertex. This vertex has factorized form and its matter part has the following expression

$$|V_3\rangle^{\text{matter}} = \int d^{26}p^{(1)}d^{26}p^{(2)}d^{26}p^{(3)}\delta(p^{(1)}+p^{(2)}+p^{(3)})\exp\left[-\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty}a_n^{(r)\dagger}V'^{rs}_{nm}a_m^{(s)\dagger}\right.$$
$$\left.-\frac{1}{\sqrt{2}}\sum_{r,s=1}^{3}\sum_{n=1}^{\infty}p^{(r)}V'^{rs}_{0n}a^{(s)\dagger}-\frac{1}{4}V'_{00}\sum_r(p^{(r)})^2\right]\bigotimes_{r=1}^{3}|p^{(r)}\rangle\,. \tag{1.1}$$

The infinite matrices V'^{rs} are called Neumann matrices. Let C be the twist matrix: $C'_{mn} = (-1)^n\delta_{mn}$. Denote

$$M'^{rs}_{nm} = (C'V'^{rs})_{nm}\,, \quad n,m = 1,2,\dots$$

The matrices M^{rs} commute with each other, have real entries and are symmetric. Their spectrum was found in [2]. The set of its eigenvectors $v_n^{(\kappa)}$ is labeled by a continuous parameter $-\infty < \kappa < \infty$. The value of this parameter is an eigenvalue of the operator

$$K_1 = L_1 - L_{-1}$$

which commutes with matrices M'^{rs}. We have

$$\sum_{n=1}^{\infty}M'^{rs}_{mn}v_n^{(\kappa)} = \mu^{rs}(\kappa)v_m^{(\kappa)}$$

where the eigenvalues $\mu^{rs}(\kappa)$ are

$$\mu^{rs}(\kappa) = \frac{1}{1+2\cosh\frac{\pi\kappa}{2}}\Big[1-2\delta_{r,s}+e^{\frac{\pi\kappa}{2}}\delta_{r+1,s}+e^{-\frac{\pi\kappa}{2}}\delta_{r,s+1}\Big]. \tag{1.2}$$

The eigenvectors $v_n^{(\kappa)}$ are given by their generating function

$$f^{(\kappa)}(z) = \sum_{n=1}^{\infty}\frac{v_n^{(\kappa)}}{\sqrt{n}}z^n = \frac{1}{\kappa\sqrt{\mathcal{N}(\kappa)}}(1-e^{-\kappa\tan^{-1}z}) \tag{1.3}$$

where

$$\mathcal{N}(\kappa) = \frac{2}{\kappa}\sinh\left(\frac{\pi\kappa}{2}\right)\,.$$

It was also shown in [2], [3] that this set of eigenvectors is orthogonal and complete

$$\sum_{n=1}^{\infty}v_n^{(\kappa)}v_n^{(\kappa')} = \delta(\kappa-\kappa')\,, \tag{1.4a}$$

$$\int_{-\infty}^{+\infty}d\kappa\, v_n^{(\kappa)}v_m^{(\kappa)} = \delta_{n,m}\,. \tag{1.4b}$$

Let us also note that the spectral representation for Neumann coefficients V'^{rs}_{0n} can be also worked out using the standard relations between these coefficients and matrices M'^{rs}. In the basis $v_n^{(\kappa)}$ these coefficients can be considered as vectors in l_∞ space and the coordinates for these vectors with respect to the basis $v_n^{(\kappa)}$ can be found (see for example [6] formulae (3.9), (3.10)). Notice that the vertex (1.1) can also be written in the alternative form, which can be obtained by integrating over momentum. In this case the vertex is specified by matrices M^{rs} whose matrix indices run from 0 to ∞. The spectral representation of these matrices was found in [4, 5]. In the present paper we will construct spectral density related to matrices M'^{rs}.

The spectral measure for spectral parameter κ was first considered in [2]. It was shown that if one truncates the matrix $(K_1)_{mn}$ to a finite $L \times L$ matrix by restricting $n, m = 1, \dots L$ the eigenvalues have a uniform distribution with density

$$\rho^L(\kappa) \sim \frac{\log L}{2\pi}$$

in the limit $L \to \infty$. There are however corrections to the density at finite L and as it was already noted in [2] these corrections become large for large $|\kappa|$ (and fixed L).

In [6] we presented a numerical study of the spectral density function that indicates that there is a nontrivial part $\rho_{\mathsf{fin}}(\kappa)$ of the spectral density that stays finite in the limit $L \to \infty$. More precisely using the orthogonality and completeness relations (1.4) one can write the following expression for the level truncated distribution function

$$\rho^L(\kappa) = \sum_{n=1}^{L} v_n^{(\kappa)} v_n^{(\kappa)} \,. \tag{1.5}$$

This expression stems from the fact that for a given symmetric matrix A_{nm} with spectral representation $A(\kappa)$ the following identity is true

$$\sum_{n=1}^{L} A_{nn} = \int_{-\infty}^{\infty} \rho^L(\kappa) d\kappa \, A(\kappa) \,. \tag{1.6}$$

One can then numerically study the difference $\rho^L(\kappa) - \frac{\log L}{2\pi}$. The numerical evidence we obtained in [6] suggests that for large L this difference converges, at least in the vicinity of $\kappa = 0$ to a well- defined continuous function. We will formally define this function as

$$\rho_{\mathsf{fin}}(\kappa) = \lim_{L\to\infty} \rho^{2L}(\kappa) - \frac{1}{2\pi} \sum_{n=1}^{L} \frac{1}{n} \tag{1.7}$$

where the limit $2L$ instead of L and the use of the sum of $1/n$ instead of the logarithm are dictated solely by technical convenience.

By using the known formula for coefficients $v_n^{(0)}$ it is easy to show analytically that $\rho_{\mathsf{fin}}(0) = \frac{\log 2}{\pi}$.

Our goal in this paper is to obtain an exact analytic expression for $\rho_{\mathsf{fin}}(\kappa)$. To achieve that goal we use a separate regularization for both terms in (1.7) defining

$$\rho^q(\kappa) = \sum_{n=1}^{\infty} v_n^{(\kappa)} q^n v_n^{(\kappa)} \tag{1.8}$$

and

$$\rho^q_{\mathsf{fin}}(\kappa) = \rho^q(\kappa) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{q^{2n}}{n} = \rho^q(\kappa) + \frac{1}{2\pi}\log(1-q^2)$$

where q is a positive number less than one. We are looking then for a limit $q \to 1$ of this expression. In fact it is technically more advantageous, but leads to the same result, if we consider the limit $q \to 1$ of the quantity

$$\rho^q_{\mathsf{fin}} = \rho^q(\kappa) - \rho^q(0) + \frac{\log 2}{\pi}\,. \tag{1.9}$$

The computation of this quantity is done in the next section by performing a certain contour integration. Taking the limit $q \to 1$ one obtains

$$\rho_{\mathsf{fin}}(\kappa) = \frac{2\log 2 - \gamma_E}{2\pi} - \frac{1}{2\pi}\left[\psi\left(1+\frac{\kappa}{2i}\right) + \psi\left(1-\frac{\kappa}{2i}\right)\right] \tag{1.10}$$

where γ_E is the Euler constant and $\psi(z)$ is the logarithmic derivative of the Γ-function. Formula (1.10) arises in computations as an integral representation

$$\rho_{\mathsf{fin}}(\kappa) = \frac{\log 2}{\pi} + \frac{1}{2\pi}\int_0^{\infty} dt\, \frac{\cos\left(\frac{\kappa t}{2}\right) - 1}{e^t - 1}\,. \tag{1.11}$$

Formulae (1.10), (1.11) are the main result of this paper. A particular SFT computation involving $\rho_{\mathsf{fin}}(\kappa)$ is a computation of an overlap of two surface states discussed in [6]. Even with the use of the exact analytic expression (1.10) this computation still seems to be at odds with CFT results [7]. We hope to return to this issue in the nearest future.

2 Computation

We need to find an analytic expression for the regulated spectral density $\rho^q(\kappa)$ defined in (1.8). Let us start by computing a more general expression

$$\rho^q(\kappa,\kappa') = \sum_{n=1}^{\infty} v_n^{(\kappa)} q^n v_n^{(\kappa')}\,.$$

It can be expressed via a contour integral (see Figure 1a)

$$\rho^q(\kappa,\kappa') = \frac{q}{2\pi i}\oint_{C_r} (\partial f)^{(\kappa)}(qz) f^{(\kappa')}\left(\frac{1}{z}\right)\,. \tag{2.1}$$

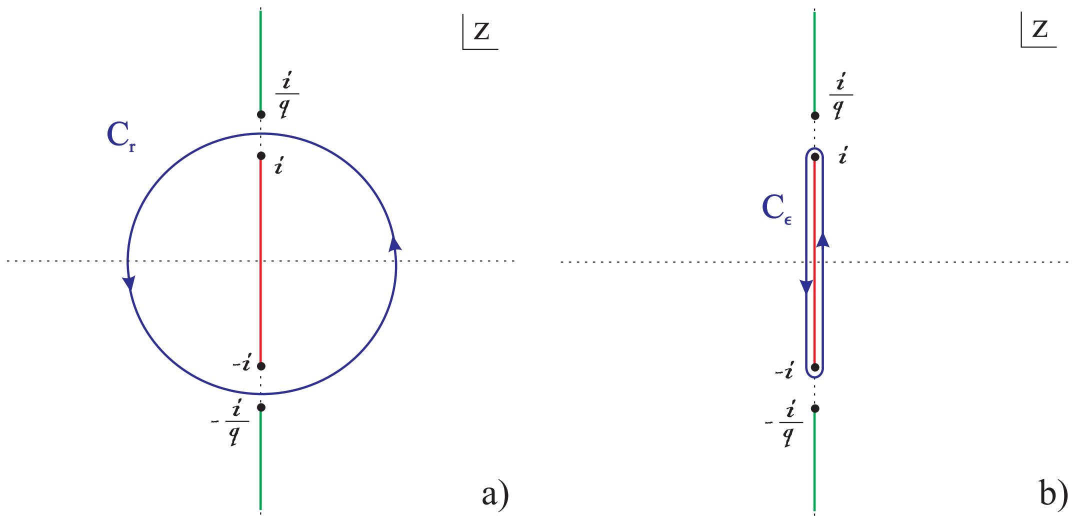

Figure 1:

Here

$$(\partial f)^{(\kappa)}(qz) = \frac{1}{\sqrt{\mathcal{N}(\kappa)}} \frac{1}{1+(qz)^2} \exp(-\kappa \tan^{-1}(qz))$$

is an analytic function defined on a complex plane with cuts going along the imaginary axis from i/q to $+i\infty$ and from $-i/q$ to $-i\infty$ and the branch of

$$\tan^{-1}(qz) = \frac{1}{2i} \log\left(\frac{1+iqz}{1-iqz}\right)$$

is chosen by the standard series expansion within the circle $|z| < 1/q$, i.e. one takes main branches for both logarithms. The function

$$f^{(\kappa')}\left(\frac{1}{z}\right) = \frac{1}{\kappa' \sqrt{\mathcal{N}(\kappa')}} \left(1 - \exp(-\kappa' \tan^{-1}(1/z))\right)$$

is defined on a complex plane with a cut going along the imaginary axis from $-i$ to i and the branch is fixed by the standard series expansion outside of the circle $|z| = 1$. The contour C_r in (2.1) is chosen to be a circle centered at the origin and of radius $1 < r < 1/q$, as both functions are holomorphic within the annulus $1 < |z| < 1/q$. The contour and the cuts are depicted on Figure 1a. We can deform the circle C_r to the contour $C_\epsilon = \{z = ix - \epsilon\} \cup \{z = ix + \epsilon\} \cup C_i \cup C_{-i}$, $-1 \le x \le 1$ depicted on Figure 1b. We obtain then

$$\begin{aligned}
\rho^q(\kappa,\kappa') &= \frac{q}{2\pi} \int_{-1}^{1} dx\, (\partial f)^{(\kappa)}(qix) \left[f^{(\kappa')}\left(\frac{1}{ix+\epsilon}\right) - f^{(\kappa')}\left(\frac{1}{ix-\epsilon}\right)\right] \\
&= \frac{q}{\pi} \sqrt{\frac{\mathcal{N}(\kappa')}{\mathcal{N}(\kappa)}} \int_{-1}^{1} \frac{dx}{1-(qx)^2} \exp\left[\frac{i\kappa}{2} \log\left(\frac{1-qx}{1+qx}\right) + \frac{i\kappa'}{2} \log\left(\frac{1+x}{1-x}\right)\right] \qquad (2.2)
\end{aligned}$$

where the last expression is evaluated in the limit $\epsilon \to 0$, we used the fact the integral over half circus of radius ϵ C_i and C_{-i} is zero and

$$\tan^{-1}\left(\frac{1}{ix \pm 0}\right) = \frac{1}{2i} \log\left(\frac{1+x}{1-x}\right) \pm \frac{\pi}{2}\,, \quad -1 \leq x \leq 1\,.$$

If at this point we make in (2.2) a substitution

$$y = \frac{1}{2} \log\left(\frac{1-qx}{1+qx}\right)$$

we arrive at the expression

$$\rho^q(\kappa, \kappa') = \frac{1}{2\pi} \sqrt{\frac{\mathcal{N}(\kappa')}{\mathcal{N}(\kappa)}} \int_{-R}^{R} dy \, \exp\big[iy(\kappa - \kappa')\big] \exp\left[\frac{i\kappa'}{2} g_q(y)\right] \tag{2.3a}$$

where

$$g_q(y) = \log\left(\frac{(q - \tanh y)\,(1 + \tanh y)}{(q + \tanh y)\,(1 - \tanh y)}\right) \tag{2.3b}$$

and

$$R = R(q) = \operatorname{arctanh}(q) \to \infty \;\text{ as }\; q \to 1\,. \tag{2.3c}$$

Using the last expression it can be shown that

$$\lim_{q \to 1} \rho^q(\kappa, \kappa') = \delta(\kappa - \kappa')\,.$$

On a formal level we note that the support of the function $g_q(y)$ is pushed to $|y| = \infty$ as $q \to 1$. On the other hand if we consider an integral $\int_{-\infty}^{\infty} d\kappa \rho^q(\kappa, \kappa') t(\kappa)$ where $t(\kappa)$ is a Schwartz class test function, then the Fourier transform $T(y)$ of the function $t(\kappa)/\sqrt{\mathcal{N}(\kappa)}$ provides a damping factor at $y \to \infty$ that allows one to drop out the function $g_q(y)$ in the limit $q \to 1$. This provides an alternative (and in our opinion a cleaner) derivation of the orthogonality property (1.4).

This being noted we set now $\kappa = \kappa'$ in (2.2) and rewrite it as

$$\rho^q(\kappa) = \frac{1}{\pi} \int_0^q \frac{dx}{1 - x^2} \cos\left[\frac{i\kappa}{2} \log\left(\frac{1 - \frac{1-q}{1+x}}{1 - \frac{1-q}{1-x}}\right)\right] \tag{2.4}$$

As the function

$$\log\left(1 - \frac{1-q}{1+x}\right)$$

uniformly converges to zero on $\mathbb{R}_+$ in the limit $q \to 1$ we can safely drop it in the above expression. After a substitution

$$t = \log\left(1 - \frac{1-q}{1-x}\right)$$

we obtain using (1.9)

$$\rho_{\mathsf{fin}}(\kappa) = \frac{\log 2}{\pi} + \frac{1}{2\pi} \lim_{q\to 1} \int_{\log q^{-1}}^{\infty} dt \, \frac{\cos\left(\frac{\kappa t}{2}\right) - 1}{e^t - 1 + \frac{q-1}{2} e^t}$$

that uniformly converges to the function (1.11). This completes the derivation of that formula.

3 Verification of the analytic expression

In this section we present two consistency checks verifying that (1.10) gives the right finite part of the spectral density.

3.1 Numerical check

Using equation (1.7) for finite value of L one can calculate the finite part of the spectral density in the vicinity of zero. On Figure 2 we present the result of this calculation for $L = 91$ (circled line). One sees that the numeric result is in a very good agreement with our analytic expression for the finite part of the spectral density (solid line). The exact analytic expression also gives the asymptotic behavior of $\rho_{\mathsf{fin}}(\kappa)$ not captured by the numerical results. Namely $\rho_{\mathsf{fin}} \sim -\frac{1}{\pi} \log|\kappa|$ as $|\kappa| \to \infty$.

3.2 Analytic check. Trace of operator B.

In [2] a certain operator denoted by B was introduced for which both the matrix elements and the eigenvalues are of a simple form and known explicitly. Namely we have

$$B_{nm} = \begin{cases} -\frac{(-1)^{\frac{m-n}{2}}\sqrt{nm}}{(m+n)^2-1}, & n+m \text{ even;} \\ 0, & n+m \text{ odd.} \end{cases}$$

while the eigenvalues corresponding to eigenvectors $v_n^{(\kappa)}$ reads

$$\beta(\kappa) = -\frac{1}{4}\frac{\pi\kappa}{\sinh\left(\frac{\pi\kappa}{2}\right)} = -\frac{\pi}{2\mathcal{N}(\kappa)}\,.$$

We can use these representations to check the asymptotic form of relation (1.6). Namely from (1.6) we have

$$\sum_{n=1}^{2L} B_{nn} = -\sum_{n=1}^{2L} \frac{n}{4n^2-1} = \int_{-\infty}^{+\infty} d\kappa\, \beta(\kappa) \left[\frac{1}{2\pi}\sum_{n=1}^{L}\frac{1}{n} + \rho_{\mathsf{fin}}^{2L}(\kappa)\right]$$

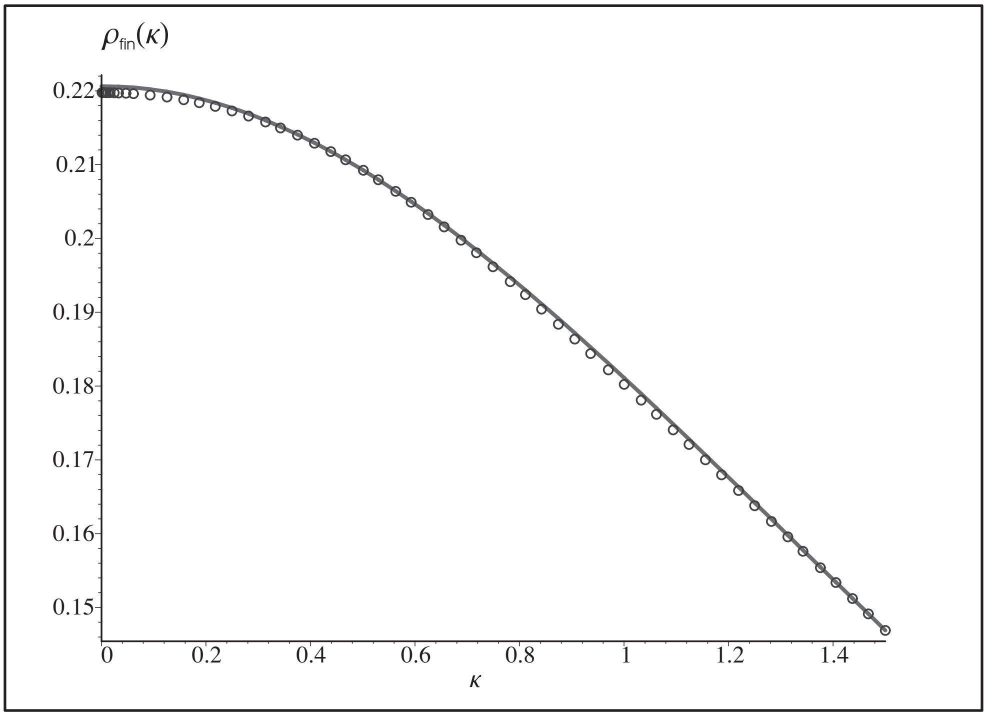

Figure 2: The solid line represents a plot of analytic expression for the finite part of the spectral density (1.11). The circles represents numerical results for the finite density obtained by evaluating (1.7) for $L = 91$.

and therefore

$$\int_{-\infty}^{+\infty} d\kappa\, \rho_{\mathsf{fin}}(\kappa)\beta(\kappa) = \lim_{L\to\infty}\left[\sum_{n=1}^{L}\frac{1}{4n} - \sum_{n=1}^{2L}\frac{n}{4n^2-1}\right] = \frac{1}{4} - \frac{3\log 2}{4}\,. \tag{3.1}$$

On the other hand if we use our analytic expression (1.11) for $\rho_{\mathsf{fin}}(\kappa)$ we have

$$\int_{-\infty}^{+\infty} d\kappa\, \rho_{\mathsf{fin}}(\kappa)\beta(\kappa) = \frac{\log 2}{\pi}\int_{-\infty}^{+\infty} d\kappa\, \beta(\kappa) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa \int_{0}^{\infty} dt\, \frac{\beta(\kappa)(\cos\left(\frac{\kappa t}{2}\right) - 1)}{e^t - 1}\,.$$

The first integral can be evaluated in a straightforward way with the result $-\frac{1}{2}\log 2$ while in the second term we can first evaluate the integral over κ by summing up residues in the upper half plane:

$$\int_{-\infty}^{+\infty} d\kappa\, \frac{e^{i\kappa t}}{\mathcal{N}(\kappa)} = \frac{1}{\cosh^2 t}$$

and then take the integral over t

$$\frac{1}{4}\int_{0}^{\infty} dt\, \frac{1 - e^{-t}}{e^t + e^{-t} + 2} = \frac{1}{4} - \frac{\log 2}{4}\,.$$

Combining both terms together we obtain the same number as in (3.1).

Note added: While this paper was nearing completion, the paper [8] appeared, which contains in Section 3 the same result (1.10) we obtained. The calculations in that paper are technically different from the ones we present.

Acknowledgments

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References

[1] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. B **268**, 253 (1986).

[2] L. Rastelli, A. Sen, B. Zwiebach, *Star Algebra Spectroscopy*, hep-th/0111281

[3] K. Okuyama, *Ghost Kinetic Operator of Vacuum String Field Theory*, hep-th/0201015

[4] B. Feng, Y.-H. He, N. Moeller, *The Spectrum of the Neumann Matrix with Zero Modes*, hep-th/0202176

[5] D. Belov, *Diagonal representation of open string star and Moyal product*, hep-th/0204164.

[6] D. M. Belov and A. Konechny, *On continuous Moyal product structure in string field theory*, hep-th/0207174.

[7] A. Le Clair, M. Peskin and C. Preitschopf, *String field theory on the conformal plane (II). Generalized gluing*, Nucl. Phys. **B317** (1989) 464.

[8] E. Fuchs, M. Kroyter and A. Marcus, *Virasoro operators in the continuous basis of string field theory*, hep-th/0210155